DEVELOPMENT OF PERTURBATIONS IN PLANE SHOCK WAVES

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Investigation of the stability of plane shock waves as regards nonuniform perturbations was first performed by D'yakov [1]. He obtained criteria for stability, and showed that perturbations grow exponentially with time in the case of instability. Iordanskii [2] has shown that in the case of stability, the perturbations are attenuated according to a power law. However, the stability criteria of [2] do not agree with the results of [1]. Kontorovich [3] has explained the cause of the apparent discrepancies, and asserts the correctness of the criteria of [2]. A power law for the attenuation of perturbations has also been obtained in [4, 5] under a somewhat different formulation of the boundary conditions.

The Cauchy problem with perturbations is examined in §1 of this paper, results are obtained for cases of practical interest, and the asymptotic behavior is investigated.

In §2 the effect of a low viscosity on the development of perturbations is examined. It is shown that when $t \to \infty$ the amplitude of perturbations is attenuated mainly as $\exp(-\alpha t)$, where $\alpha > 0$ does not depend on the form of the boundary conditions at the shock wave front. The results of §2 were used in processing the experimental data of [6], which made it possible to determine the viscosity of a number of substances at high pressure.

§1. Let a plane shock wave move with constant speed v_0 in a uniform material occupying the whole of space. We shall choose a system of coordinates in which the wave front is at rest and coincides with yz-plane. The x-axis is along the direction of motion of the gas. Ahead of the wave the gas has pressure p_0, density ρ_0, velocity of sound c_0, and mass velocity v_0. The corresponding values behind the front are p, ρ, c, and v. Since $v_0 > c_0$, perturbation localized in a finite region ahead of the wave front will be absorbed by it in a finite time. By choosing this instant as the origin of time, we can assume that there are no perturbations ahead of the wave. Without loss of generality, we confine ourselves to the case in which all quantities are independent of z, and the velocity component along the z-axis is zero. The system of equations for small perturbations, denoted by primes, has the form

$$\frac{\partial v_x'}{\partial t} + v\frac{\partial v_x'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0,$$

$$\frac{\partial v_y'}{\partial t} + v\frac{\partial v_y'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial y} = 0,$$

$$\frac{\partial p'}{\partial t} + v\frac{\partial p'}{\partial x} + \rho c^2\left(\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y}\right) = 0. \qquad (1.1)$$

The density perturbation ρ' may be eliminated by means of the condition for adiabatic flow:

$$\frac{\partial \rho'}{\partial t} + v\frac{\partial \rho'}{\partial x} = \frac{1}{c^2}\left(\frac{\partial p'}{\partial t} + v\frac{\partial p'}{\partial x}\right). \qquad (1.2)$$

We shall solve the problem under the assumption that all the quantities depend on y as $\exp(ik_0y)$. We introduce the notation

$$\delta = -j^2\left(\frac{\partial V}{\partial p}\right)_H, \qquad (1.3)$$

where $j = \rho_0 v_0 = \rho v$ is the mass flux density through the front, and the derivative of the specific volume $V = 1/\rho$ with respect to pressure p is calculated along the shock adiabat. For an ideal gas $\delta = 1/M_0^2$, where $M_0 = v_0/c_0$ is the Mach number, determined as the ratio of the wave speed through the cold material to the speed of sound in it. Let $\xi(y, t)$ be the displacement of the shock wave front from the plane $x = 0$, where $\xi > 0$ if the front is swept away towards the compressed gas side. The conditions at the wave front obtained in [1] may be written at $x = 0$ as

$$(v_0 - v)\frac{\partial \xi}{\partial y} = v_y', \quad v_x' + \frac{1+\delta}{2\rho v}p' = 0,$$

$$\frac{\partial \xi}{\partial t} + \frac{1-\delta}{2\rho_0(v_0 - v)}p' = 0. \qquad (1.4)$$

We assign the initial conditions at $t = 0$ in the form

$$p' = \rho v^2 f_1(x)e^{ik_0y}, \qquad v_x' = vf_2(x)e^{ik_0y},$$

$$v_y' = -ivf_3(x)e^{ik_0y}, \qquad \xi = \xi\, e^{ik_0y}. \qquad (1.5)$$

For the initial data not to contradict the boundary conditions, the functions $f_i(x)$ (i = 1, 2, 3) must satisfy relations which can be obtained from (1.1) and (1.4):

$$f_2(0) + \frac{1+\delta}{2}f_3(0) = 0,$$

$$\left(\frac{df_3}{dx}\right)_{x=0} + k_0\left(\frac{1-\delta}{2}\frac{\rho}{\rho_0} - 1\right)f_1(0) = 0. \qquad (1.6)$$

In addition, we assume that there is no source of perturbations at great distances from the wave front, and therefore for finite time values all perturbations must go to zero as $x \to +\infty$. The Cauchy problem for system (1.1) is easily solved by means of the Laplace transformation. To this end we transform from an original $f(x, t)$ to a transform $f(x, s)$:

$$f(x, t) \div f(x, s) = \int_0^\infty f(x, t)e^{-st}\,dt. \qquad (1.7)$$

For brevity we omit the factor $\exp(ik_0y)$ which gives rise to no difficulties. In lieu of (1.1) we obtain for the transforms the system

$$sv_x' + v\frac{dv_x'}{dx} + \frac{1}{\rho}\frac{dp'}{dx} = vf_2(x),$$

$$sv_y' + v\frac{dv_y'}{dx} + \frac{ik_0}{\rho}p' = -ivf_3(x),$$

$$sp' + v\frac{dp'}{dx} + \rho c^2\left(\frac{dv_x'}{dx} + ik_0v_y'\right) = \rho v^2 f_1(x). \qquad (1.8)$$

Using (1.4), we obtain boundary conditions for the transforms at $x = 0$:

$$ik_0(v_0 - v)\xi = v_y', \quad v_x' + \frac{1+\delta}{2\rho v}p' = 0,$$

$$s\xi + \frac{1-\delta}{2\rho_0(v_0 - v)}p' = \xi_0. \qquad (1.9)$$

Moreover, when $x \to +\infty$, all the transformed functions must go to zero, of Re $s > s_0$, where s_0 is a sufficiently small positive number.

Applying the method of variation of constants and returning to the original functions, we obtain the formula for the amplitude of displacement of the front

$$\xi(\tau) = \frac{1}{2\pi i} \int \frac{e^{\tau z}}{D(z)} \Big\{ \xi_0 [(1 + \delta - 2\beta^2) z^2 + (1 - \delta) z\omega - 2(1 - \beta^2)] - \frac{\sigma}{\sigma - 1}(1 - \delta) F(z) \Big\} dz,$$

$$\tau = k_0 vt, \quad \sigma = \rho/\rho_0, \quad \beta = v/c < 1,$$

$$z = s/k_0 v, \quad \omega = \sqrt{\beta^2 z^2 + 1 - \beta^2}. \qquad (1.10)$$

The integral is taken along the vertical straight line lying to the right of all singularities of the subintegral expression, the branch being chosen which for positive numbers will give an arithmetic value of the radical. The function F(z) is determined by the profile of the initial perturbations

$$F(z) = \int_0^\infty [\beta^2 (z^2 - 1) f_1(x) + (1 - \omega z) f_2(x) + (\omega - z) f_3(x)] \exp\left(-\frac{\beta^2 z + \omega}{1 - \beta^2} k_0 x\right) dx. \qquad (1.11)$$

The denominator of the subintegral expression in (1.10) contains the function

$$D(z) = (1 + \delta - 2\beta^2) z^3 + [\sigma(1 - \delta) - 2(1 - \beta^2)] z + (1 - \delta)(z - \sigma)\sqrt{\beta^2 z^2 + 1 - \beta^2}. \qquad (1.12)$$

The location of the zeros of D(z) in the plane of the complex variable z depends on three dimensionless parameters δ, σ, β. It follows from (1.11) that the function F(z) does not go to infinity for any finite z in the right half-plane. It is therefore sufficient, in an investigation of stability, to examine the special case in which only the shock wave front is distorted at time zero, and there were no other perturbations for $x > 0$:

$$f_1(x) = f_2(x) = f_3(x) = 0. \qquad (1.13)$$

We note that these were exactly the initial conditions in the tests in [6]. Then F(z) = 0, and (1.10) takes the form

$$\varphi(\tau) = \frac{\xi(\tau)}{\xi_0} = \frac{1}{2\pi i} \int \frac{e^{\tau z}}{D(z)} [(1 + \delta - 2\beta^2) z^2 + (1 - \delta) z\omega - 2(1 - \beta^2)] dz. \qquad (1.14)$$

Converting to the new complex variable w,

$$z = {}^1/_2 \mu (w - w^{-1}), \; w = z/\mu + [(z/\mu)^2 + 1]^{1/2},$$

$$\mu = \sqrt{\varepsilon}, \quad \varepsilon = (1 - \beta^2)/\beta^2, \qquad (1.15)$$

we obtain

$$\varphi(T) = \frac{1}{2\pi i} \int \exp\left[\frac{1}{2} T\left(w - \frac{1}{w}\right)\right] \times \frac{(1 + \delta + 2\beta) w^2 - (1 + \delta - 2\beta)}{f(w^2)} dw, \qquad (1.16)$$

$$f(x) = (1 + \delta + 2\beta) x^2 + [4\sigma\varepsilon^{-1}(1 - \delta) - 2(1 + \delta)] x + 1 + \delta - 2\beta. \qquad (1.17)$$

The argument T is connected with τ by the relation

$$T = \mu\tau = k_0 ct \sqrt{1 - \beta^2}. \qquad (1.18)$$

The velocity of propagation of a small perturbation along the surface of the shock wave front is equal to $(c^2 - v^2)^{1/2} = c(1 - \beta^2)^{1/2}$. Therefore the quantity T has the meaning of the distance traversed by the signal along the wave front surface, measured in the units $\lambda' = \lambda/2\pi = 1/k_0$, where λ is the wavelength of the perturbations.

Because of the properties of the transformation (1.15), the integral in (1.16) must also be calculated along the vertical straight line located to the right of all singularities of the subintegral function.

It may be shown [1–3] that if

$$\frac{\sigma - \varepsilon}{\sigma + \varepsilon} < \delta < 1, \qquad (1.19)$$

then both of the zeros x_1 and x_2 of the function $f(x)$ lie within the circle $|x| = 1$. The subintegral expression in (1.16) satisfies the conditions of Jordan's lemma [7], and therefore, in the case of (1.19), the integral in (1.16) may be taken along the unit circle $|w| = 1$. Carrying out the necessary calculations, we obtain

$$\varphi(T) = \frac{4}{\pi} \beta\varepsilon\sigma(1 - \delta) \int_0^1 \cos(Tx) \frac{\sqrt{1 - x^2}\, dx}{A\varepsilon^2 x^4 + 2B\varepsilon x^2 + C}, \qquad (1.20)$$

$$A = (1 + \delta)^2, \quad B = 2(1 - \beta^2) - \sigma(1 - \delta^2),$$

$$C = \sigma^2 (1 - \delta)^2. \qquad (1.21)$$

The denominator in (1.20) does not go to zero in the entire segment $0 \le x \le 1$ if condition (1.19) is satisfied. It is clear that the function $\varphi(T)$, determined by (1.20), tends to zero when $T \to \infty$. Therefore (1.19) ensures the stability of the shock wave. Because of (1.15) the neighborhoods $|w| = 1$ correspond in the z plane to two branch sections in the segment $(-i\mu, i\mu)$ of the imaginary axis. The ends of the section are the branch points of the function $\omega = (\beta^2 z^2 + 1 - \beta^2)^{1/2}$ The region $|w| > 1$ is mapped on the lower sheet of the Riemann surface on which the contour of integration in (1.14) lies. The region $|w| < 1$ maps onto the upper sheet. The two sheets join along the segment $(-i\mu, i\mu)$ of the imaginary axis of the z-plane. Thus, in the stable case, the asymptotic behavior of the perturbations is determined by the location of the branch points of the subintegral expression in (1.14); the poles lying on the other sheet do not contribute to the asymptote.

We note here, as follows from [1–3], that when one of the conditions $\delta > 1$ or $\delta < -(1 + 2\beta)$ is satisfied, the zeroes of $f(x)$ satisfy the inequalities $x_1 > 1$ and $-1 < x_2 < 1$. It follows from (1.16) that there is then an exponential growth of the perturbations (absolute instability). If

$$-(1 + 2\beta) < \delta < (\sigma - \varepsilon)/(\sigma + \varepsilon), \qquad (1.22)$$

then $x_1 < -1$, $-1 < x_2 < 1$. It was shown in [1-3] that then the front radiates sound waves, and the surface of the front oscillates according to a harmonic law.

The asymptotic behavior of $\varphi(T)$ is most simply obtained from (1.11) by the method of steepest descents as

$$\varphi(T\to\infty) \sim \frac{4\beta\varepsilon\sigma(1-\delta)}{[(1+\delta)\varepsilon-\sigma(1-\delta)]^2}\frac{\sin(T-{}^1/_4\pi)}{\sqrt{2\pi T^3}}. \quad (1.23)$$

If the material through which the shock wave is propagating is an ideal gas with isentropic index γ, (1.23) takes the form

$$\varphi(T\to\infty) \sim \frac{4}{\delta^2}\left(\frac{1+(h-1)\delta}{h+1-\delta}\right)^{1/2}\frac{\sin(T-{}^1/_4\pi)}{\sqrt{2\pi T^3}}$$

$$\left(h=\frac{\gamma+1}{\gamma-1}\right). \quad (1.24)$$

For a strong shock wave ($\delta = 0$), (1.24) diverges. This points to the fact that the asymptotic behavior of $\varphi(T)$ for a strong shock will be substantially different; the attenuation will be slower. The exact formula (1.20) for a strong shock may be written as

$$\varphi(T) = \frac{4}{\pi}\sqrt{h+1}\int_0^1 \frac{\cos(Tx)\,dx}{[h+1-(h-3)x^2]\sqrt{1-x^2}}. \quad (1.25)$$

Hence we find that in the strong shock case

$$\varphi(T\to\infty) \sim \left(\frac{h+1}{2\pi T}\right)^{1/2}\cos\left(T-\frac{\pi}{4}\right). \quad (1.26)$$

It is curious that for $\gamma = 2$ ($h = 3$), (1.25) takes the simple form

$$\varphi(T) = J_0(T), \quad (1.27)$$

where J_0 is a Bessel function. This is evidently due to the fact that in this case $u = v_0 - v = c$.

In the ideal case in which $\delta \ll 1$ and $T \gg 1$, the behavior of $\varphi(T)$ depends on the relation between δ and T. In a similar manner to what was done in [4, 5], we find that when $\delta \ll 1$ and $T \gg 1$, the true formula is

$$\varphi(T) \sim \sqrt{\frac{h+1}{2\pi T}}\left\{\cos\left(T-\frac{\pi}{4}\right) - \sqrt{\pi q}\cos(T+q) + 2\sqrt{q}\left[S(\sqrt{q})\cos\left(T+q-\frac{\pi}{4}\right) - C(\sqrt{q})\sin\left(T+q-\frac{\pi}{4}\right)\right]\right\},$$

$$S(z) = \int_0^z \sin(t^2)\,dt, \qquad C(z) = \int_0^z \cos(t^2)\,dt,$$

$$q = {}^1/_8(h+1)T\delta^2. \quad (1.28)$$

Here S(z) and C(z) are Fresnel integrals.

Suppose that a strong enough wave ($\delta \ll 1$) has traversed a large distance, such that $T \gg 1$. If δ is so small that $q \ll 1$, then in this phase there will be attenuation of the perturbations according to the strong wave law (1.26). With increase of T the parameter q increases. When $q \sim 1$, there will be a noticeable deviation from (1.26), although attenuation of the perturbations will continue. At this phase of the asymptote we should use the more general formula (1.28). For still larger values of T, when the parameter q becomes large enough: $q \gg\gg 1$, and attenuation of the perturbations takes place according to the law (1.24).

In general, in the presence of initial perturbations distributed throughout the space, the displacement of the shock wave front is described by the formula

$$\xi(T) = \xi_0\varphi(T) + \psi(T), \quad (1.29)$$

where, in accordance with (1.10),

$$\psi(T) = -\frac{\sigma(1-\delta)}{\sigma-1}\frac{1}{2\pi i}\int\frac{F(z)}{D(z)}\exp\left(\frac{Tz}{\mu}\right)dz. \quad (1.30)$$

The asymptotic behavior of the functions $\varphi(T)$ and $\psi(T)$ is qualitatively the same, but we think that the formulas for $\psi(T)$ are more cumbersome. We shall bring in the asymptotic formula for $\psi(T)$ only in the case of a wave in an ideal gas:

$$\psi(T) \sim \frac{1}{h-1}\left(\frac{h+1}{2\pi T}\right)^{1/2}\mathrm{Re}\int_0^\infty [f_2(x) - f_1(x) - i\sqrt{h}f_3(x)]\exp\left[i\left(T-\frac{\pi}{4}-\frac{k_0x}{\sqrt{h}}\right)\right]dx. \quad (1.31)$$

§2. The relation for $\varphi(T)$ obtained in §1 (see formula (1.20)) was checked by experiment [6]. It was noted in the course of the experiment that changing the linear dimensions while retaining geometric similarity did not lead to complete coincidence of the curves in (φ, T) coordinates. This means that an appreciable role is played by such nonmodeled parameters as the viscosity of the compressed material. It turned out, however, that the curves of $\varphi(T)$ observed in the similar transformation of the system did not differ appreciably. Therefore the viscous terms may be considered as a correction in the equations of motion. We assume that the material through which the shock wave has passed has been stripped of its solidity. We assume also that the thermal conductivity and the second viscosity coefficient are zero. In the undisturbed flow behind the wave front all the quantities are independent of the coordinates and time, and so the condition for adiabatic flow is not changed in the linear approximation. When viscosity is present we have, in lieu of (1.1), the system of equations

$$\frac{\partial v_x'}{\partial t} + v\frac{\partial v_x'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = \nu\left(\frac{4}{3}\frac{\partial^2 v_x'}{\partial x^2} + \frac{\partial^2 v_x'}{\partial y^2} + \frac{1}{3}\frac{\partial^2 v_y'}{\partial x\partial y}\right),$$

$$\frac{\partial v_y'}{\partial t} + v\frac{\partial v_y'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial y} = \nu\left(\frac{1}{3}\frac{\partial^2 v_x'}{\partial x\partial y} + \frac{\partial^2 v_y'}{\partial x^2} + \frac{4}{3}\frac{\partial^2 v_y'}{\partial y^2}\right),$$

$$\frac{\partial p'}{\partial t} + v\frac{\partial p'}{\partial x} + \rho c^2\left(\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y}\right) = 0. \quad (2.1)$$

As usual, ν is the kinematic viscosity. We shall examine the order of smallness of the quantities appearing here. If the distortion of the wave front is described by an amplitude ξ and wavelength λ, the perturbations are of order ξ/λ with respect to the corresponding unperturbed quantities. In system (2.1) we neglected terms of order $(\xi/\lambda)^2$. The ratio of the right sides of (2.1) to the left sides as regards order of magnitude is $\nu/(v\lambda) = 1/R$, where R is the Reynolds number. The proposed method of allowing for viscosity rests on the fact that the right sides of (2.1) are considered to be small:

$$1/R = \nu/(v\lambda) \ll 1. \quad (2.2)$$

If, in addition,

$$\xi/\lambda \ll 1, \quad (2.3)$$

the viscous and nonlinear terms may be considered as corrections whose ratio to one another may be arbitrary. The method of successive approximations may be used to calculate each of these corrections.

Under complete geometric similarity the displacement of the curve $\varphi(T)$ is due only to viscosity, and therefore only corrections due to viscosity are considered in what follows. We note that condition (2.2) is satisfied if, for given ν and v, we take large enough λ, while condition (2.3) will be satisfied if, for a chosen λ, we take small enough ξ.

In the derivation of the boundary conditions at the shock wave front, we should take into account, in the perturbed region, the fluxes of momentum and energy due to viscosity; for x = 0

$$v_y' - (v_0 - v)\frac{\partial \xi}{\partial y} = \frac{\nu}{v}\left(\frac{\partial v_x'}{\partial y} + \frac{\partial v_y'}{\partial x}\right),$$

$$\frac{\partial \xi}{\partial t} + \frac{1-\delta}{2\rho_0(v_0 - v)}p' =$$

$$= \frac{\nu}{v}\frac{\sigma(1-\delta)}{3(\sigma-1)(1-\beta^2)}\left(2\frac{\partial v_x'}{\partial x} - \frac{\partial v_y'}{\partial y}\right),$$

$$v_x' + \frac{1+\delta}{2\rho_0 v_0}p' = \frac{\nu}{v}\frac{1+\delta-2\beta^2}{3(1-\beta^2)}\left(2\frac{\partial v_x'}{\partial x} - \frac{\partial v_y'}{\partial y}\right). \qquad (2.4)$$

Since the shock wave front has a finite width a, relations (2.4) will be true only when the dimensions of the transition zone are small in comparison with the radius of curvature r of the wave front ($a \ll r$). For a sinusoid with wavelength λ and amplitude ξ (such that $\xi^2/\lambda \ll 1$), the least radius of curvature is equal to λ^2/ξ in order of magnitude, so that the condition $a \ll r$ should be replaced by the inequality $a \ll \lambda^2/\xi$.

The width of the transition zone depends on the state of the material behind the wave front, for example, on the pressure: $a = a(p)$. For a change in pressure of amount p', the width of the front changes by amount $a' = p'da/dp$. The function $a(p)$ is smooth enough, so that the derivative $d\ln a/d\ln p$ can be regarded as a quantity of order unity. Then, as regards order of magnitude,

$$a'/a \sim p'/p \sim \xi/\lambda. \qquad (2.5)$$

In order to be able to follow the displacement of the front during a change in pressure, the condition $a' \ll \xi$ must be satisfied. Because of (2.5) this inequality may be replaced by $a \ll \lambda$. It is evident that the condition $a \ll \lambda$ ensures fulfillment of the inequality $a \ll \lambda^2/\xi$. In the tests described in [6], conditions (2.2), (2.3), and $a \ll \lambda$ were satisfied with good accuracy.

It was noted in a recent paper [8] that (2.4) should contain terms of order R^{-1} (with respect to the left sides), to take account of the shock wave structure. The authors of [8], following the method of Zhermen and Guiraud, denoted these corrections by $R^{-1}f^*$, where f^* is the effective value of some quantity f (pressure, velocity, etc.) in the transition zone. The quantity f^* was calculated in [8] on the assumption that the Navier-Stokes equation describes the shock wave structure. However, the equations of hydrodynamics are not applicable in this region. Moreover, the quantity f^* cannot be calculated using the more rigorous kinetic equation, not only while the viscosity and thermal conductivity are unknown as functions of pressure and temperature, but also while the types of interactions between molecules of the material are unknown.

In these conditions an important point is that the viscosity increases sharply with pressure. This strong dependence of viscosity on pressure leads to the fact that the width of the transition zone, a, will be small in comparison with $\nu/v = \lambda/R$, where ν is the viscosity of the compressed material far from the wave front. This is confirmed by experiments [9], in which reflection of light front the front of the shock wave was studied. Terms of type f^* cannot be taken into account in the boundary conditions (2.4) in this case, since their contribution will be of order av/ν.

Under the above assumptions, we shall write any quantity f in the form $f + f_1$, where f gives the solution to the problem with $\nu = 0$, and f_1 is a correction due to viscosity. We can substitute quantities of type f, which we consider known (see §1) into the right sides of (2.1) and (2.4), and the left sides have only the desired quantities of type f_1, since the quantities of type f, calculated without allowance for viscosity, satisfy the same scheme of linear equations, but without the right sides. The initial conditions for all the quantities of type f_1 will be zero, since the viscosity cannot manifest itself at that time.

In practice, perturbations were created at the instant a shock wave passes the joint between two large specimens made of the same material. One of the specimens had cylindrical grooves with a sinusoidal profile at the jointing surface. The depth of the grooves was small compared with the period of the sinusoid. This ensured the relative smallness of the perturbations. At the instant at which the shock wave passed the boundary between the specimens, all the perturbations were concentrated into a narrow zone, of which the thickness is of the order of the depth of the grooves. Since the perturbations themselves are of first order, the function $\psi(T)$ (see (1.30)) will be of second order. Therefore further calculations were carried out for the case when only the shock wave front was distorted at t = 0. Then the amplitude ξ_1 of the additional displacement $\xi_1\exp(ik_0y)$ of the wave front due to viscosity, following elementary but quite lengthy computations, may be represented in the form

$$\frac{\xi_1}{\xi_0} = -\frac{\nu k_0}{v}\sigma(1-\delta)\frac{1}{2\pi i}\times$$

$$\times\int e^{\tau z}\left[\frac{q(z)}{P(z)} + 2z\sqrt{\beta^2 z^2 + 1 - \beta^2}\,\frac{1+Q(z)}{P(z)}\right]dz,$$

$$P(z) = Az^4 - 2Bz^2 + C, \quad Q(z) = \frac{A_1 z^4 - 2B_1 z^2 + C_1}{3(1-\beta^2)P(z)}. \qquad (2.6)$$

Here q(z) is a polynomial of the third degree, whose exact form proves to be unimportant. The constants A, B, and C are determined by (1.21) to be

$$A_1 = -8\beta^4 + 2\beta^2(1+\delta)(11+5\delta) - 3(1+\delta)^3,$$

$$B_1 = 2\beta^2(1-\delta) - 6(1+\delta) - 4\beta^4 +$$

$$+ \sigma(1-\delta)[3(1+\delta)^2 - 4\beta^2(4+\delta)],$$

$$C_1 = \sigma(1-\delta)\{8(3-\beta^2) -$$

$$- \sigma(1-\delta)[2\beta^2 + 3(1+\delta)]\}. \qquad (2.7)$$

In deriving (2.6), the numerators and denominators of the fractions were multiplied by the quantity $D^*(z)$, obtained from (1.12) by change of the sign of the radical. In particular, $P(z) = D(z)D^*(z)$. Therefore, the subintegral expressions in (2.6) and (1.14) have the same singular points. When the conditions of stability (1.19) are satisfied (only this case is met in practice), the integral in (2.6) may be evaluated along both branch sections performed on the segment $(-i\mu, i\mu)$ of the imaginary axis of the z-plane. The first term, containing q(z), gives zero. By introducing the new variable $x = z/i\mu$, we obtain

$$\varphi_1(T) = \frac{\xi_1}{\xi_0} = \frac{\nu k_0}{v}\frac{4}{\pi}\sigma\varepsilon(1-\delta)\times$$

$$\times\sqrt{1-\beta^2}\int_0^1 \sin(Tx)\,\frac{1+Q(x)}{P(x)}\,x\sqrt{1-x^2}\,dx,$$

$$P(x) = A\varepsilon^2 x^4 + 2B\varepsilon x^2 + C,$$

$$Q(x) = \frac{A_1\varepsilon^2 x^4 + 2B_1\varepsilon x^2 + C_1}{3(1-\beta^2)P(x)}. \qquad (2.8)$$

The argument T was determined by (1.18).

The total displacement of the shock wave front, referred to its initial value ξ_0, is equal to $\varphi(T) + \varphi_1(T)$, where $\varphi(T)$ is determined by (1.20). It is assumed that φ_1 is small in comparison with φ, and therefore, where $d\varphi/dT \neq 0$, the presence of a small correction to the function $\varphi(T)$ may be regarded as a change of phase

$$\varphi(T) + \varphi_1(T) = \varphi(T - \theta),$$

$$\theta = -\varphi_1(T)(d\varphi/dT)^{-1}. \qquad (2.9)$$

Separating out the dimensionless factor containing the viscosity, we represent the phase shift in the form

$$\theta = \frac{\nu}{\lambda v_0} S \qquad \left(\lambda = \frac{2\pi}{k_0}\right). \tag{2.10}$$

Here λ is the wavelength of the perturbations, and v_0 is the speed of the shock wave in the cold material.

The quantity S, which depends on time, may be written, using (1.20) and (2.8), as follows:

$$S(T) = 2\pi\sigma\mu \int_0^1 \sin(Tx)\,[1 + Q(x)]\,x\sqrt{1 - x^2}\,\frac{dx}{P(x)} \times$$
$$\times \left[\int_0^1 \sin(Tx)\,x\sqrt{1 - x^2}\,\frac{dx}{P(x)}\right]^{-1}. \tag{2.11}$$

Formulas (1.20), (2.8), and (2.11) were used in processing the experimental data of [6].

We shall turn to examination of the asymptotic law of attenuation of the perturbations in the presence of viscosity. It is natural to expect that if the attenuation follows a power law with $\nu = 0$, then with $\nu \neq 0$ there will be a stronger exponential attenuation law.

We shall go over to the transforms in system (2.1), as was done in formulating system (1.18); without loss of generality we may restrict examination to the case when there are no initial perturbations distributed throughout the space. As before, we assume that all the quantities depend on the coordinate y via the factor $\exp(ik_0 y)$, The system of equations for the transforms takes the form

$$sv_x' + v\frac{dv_x'}{dx} + \frac{1}{\rho}\frac{dp'}{dx} =$$
$$= \nu\left(\frac{4}{3}\frac{d^2v'_x}{dx^2} - k_0^2 v_x' + \frac{ik_0}{3}\frac{dv_y'}{dx}\right),$$
$$sv_y' + v\frac{dv_y'}{dx} + \frac{ik_0}{\rho}p' =$$
$$= \nu\left(\frac{ik_0}{3}\frac{dv_x'}{dx} + \frac{d^2v_y'}{dx^2} - \frac{4}{3}k_0^2 v_y'\right),$$
$$sp' + v\frac{dp'}{dx} + \rho c^2\left(\frac{dv_x'}{dx} + ik_0 v_y'\right) = 0. \tag{2.12}$$

We shall solve the system (1.8) for $f_1 = f_2 = f_3 = 0$ with respect to the derivatives

$$\frac{dv_x'}{dx} = \frac{k_0}{1 - \beta^2}\left[\beta^2 z\left(v_x' - \frac{p'}{\rho v}\right) - iv_y'\right],$$
$$\frac{dv_y'}{dx} = -k_0\left(\frac{ip'}{\rho v} + zv_y'\right),$$
$$\frac{1}{\rho v}\frac{dp'}{dx} = \frac{k_0}{1 - \beta^2} \times$$
$$\times\left[z\left(\beta^2\frac{p'}{\rho v} - v_x'\right) + iv_y'\right] \quad \left(z = \frac{s}{k_0 v}\right). \tag{2.13}$$

Differentiating this zeroth approximation with respect to x, we substitute the expressions obtained for the second derivatives into the right sides of (2.12), which contain a small viscosity in the form of the factor

$$sv_x' + v\frac{dv_x'}{dx} + \frac{1}{\rho}\frac{dp'}{dx} =$$
$$= \nu k_0\left[\frac{4\beta^2 z}{3(1 - \beta^2)}\left(\frac{dv_x'}{dx} - \frac{1}{\rho v}\frac{dp'}{dx}\right) - \right.$$
$$\left. - k_0 v_x' - \frac{i(3 + \beta^2)}{3(1 - \beta^2)}\frac{dv_y'}{dx}\right],$$
$$sv_y' + v\frac{dv_y'}{dx} + \frac{ik_0}{\rho}p' =$$
$$= \nu k_0\left[\frac{i}{3}\frac{dv_x'}{dx} - \frac{4}{3}k_0 v_y' - \frac{i}{\rho v}\frac{dp'}{dx} - z\frac{dv_y'}{dx}\right],$$
$$sp' + v\frac{dp'}{dx} + \rho c^2\left(\frac{dv_x'}{dx} + ik_0 v_y'\right) = 0. \tag{2.14}$$

The solution of this system has the form

$$\frac{p'}{\rho v} = Ae^{\alpha k_0 x}, \qquad v_x' = Be^{\alpha k_0 x}, \qquad v_y' = Ce^{\alpha k_0 x}. \tag{2.15}$$

For the amplitudes A, B, and C we obtain the algebraic system

$$\left[\alpha + \frac{1}{R}\cdot\frac{4\beta^2 z\alpha}{3(1 - \beta^2)}\right]A +$$
$$+ \left[z + \alpha + \frac{1}{R}\left(1 - \frac{4}{3}\frac{\beta^2 z\alpha}{1 - \beta^2}\right)\right]B +$$
$$+ \frac{1}{R}\frac{(3 + \beta^2)\alpha}{3(1 - \beta^2)}iC = 0,$$
$$-\left(1 + \frac{\alpha}{R}\right)A + \frac{1}{R}\frac{\alpha}{3}B +$$
$$+ \left[z + \alpha + \frac{1}{R}\left(\frac{4}{3} + z\alpha\right)\right]iC = 0,$$
$$\beta^2(z + \alpha)A + \alpha B + iC = 0 \qquad \left(R = \frac{v}{\nu k_0}\right). \tag{2.16}$$

Here R is the Reynolds number.

Eliminating the amplitude C from the first two equations with the aid of the third, we obtain

$$\left\{\alpha + \frac{1}{R}\left[\frac{1}{3}\beta^2 z\alpha - \frac{(3 + \beta^2)\beta^2}{3(1 - \beta^2)}\alpha^2\right]\right\}A +$$
$$+ \left[z + \alpha + \frac{1}{R}(1 + \alpha z - \frac{1}{R}\frac{3 + \beta^2}{3(1 - \beta^2)}\alpha(z + \alpha)\right]B = 0,$$
$$\left\{1 + \beta^2(z + \alpha)^2 + R^{-1}\left[\alpha + \beta^2(z + \alpha)\left(\frac{4}{3} + z\alpha\right)\right]\right\}A +$$
$$+ \alpha\,[z + \alpha + R^{-1}(1 + \alpha z)]\,B = 0. \tag{2.17}$$

It being our intention to obtain the result only with accuracy up to R^{-1} inclusive, we reduce the coefficient of B in the first equation of (2.17) by

$$\frac{1}{R^2}\frac{3 + \beta^2}{3(1 - \beta^2)}\alpha(1 + \alpha z).$$

The determinant of system (2.17) is then decomposed into factors, one of which has the form $z + \alpha + R^{-1}(1 + \alpha z)$ and is unsuitable for evaluation of the asymptote. Equating the second factor to zero, we obtain the quadratic equation

$$a\alpha^2 - 2b\alpha - c = 0, \tag{2.18}$$
$$a = 1 - \beta^2 + \frac{4}{3R}\frac{\beta^2(1 + \beta^2)z}{1 - \beta^2},$$
$$b = \beta^2\left[z - \frac{2}{3R}\frac{\beta^2(z^2 + 1)}{1 - \beta^2}\right], \quad c = 1 + \beta^2 z^2 + \frac{4}{3R}\beta^2 z. \tag{2.19}$$

We find a new branch point by equating the discriminant $b^2 + ac$ of (2.18) to zero. From the equation obtained, we find, with the required accuracy, that when $\nu \neq 0$, the position of the branch point in the z-plane is given by

$$z^* = \pm i\mu - \frac{2}{3(1 - \beta^2)}\frac{\nu k_0}{v}. \tag{2.20}$$

The presence here of a negative correction leads to the fact that the absolute magnitude of the perturbations attenuates basically according to the law

$$F(t) = \exp\left(-\frac{2}{3(1 - \beta^2)}k_0^2 \nu t\right). \tag{2.21}$$

This result has a simple physical meaning.

We find from system (1.18), with zero right sides, that one of the solutions will be the sound wave $\exp(z\tau + ik_0y + ikx)$, where

$$ik = k_0 \frac{\beta^2 z - \sqrt{\beta^2 z^2 + 1 - \beta^2}}{1 - \beta^2}. \quad (2.22)$$

The wave vector $\vec{k}$ with components

$$k_x = -k_0/\mu, \; k_y = k_0 \quad (2.23)$$

corresponds to the point $z = -i\mu$.

In the coordinate system in which the compressed material is at rest, the sound frequency Ω, the modulus $k = (k_x^2 + k_y^2)^{1/2}$ of the wave vector, and the velocity of sound c are connected by the relation $\Omega = ck$.

Using (2.23) we find

$$k^2 = \frac{k_0^2}{1 - \beta^2} = \frac{\Omega^2}{c^2}. \quad (2.24)$$

Sound with wave vector (2.23) propagates along the front. In fact, the velocity of the sound wave relative to the front is

$$u = v + ck_x/k. \quad (2.25)$$

By substituting here from (2.23) and (2.24), we find that $u = 0$. Of course, the attenuation of waves whose direction is close to the above also determines the asymptotic behavior of perturbations at the wave front. Using (2.24), (2.21) may be rewritten as:

$$F(t) = \exp\left\{-\frac{2}{3}\nu\frac{\Omega^2}{c^2}t\right\}. \quad (2.26)$$

We shall compare this result with the general law for attenuation of sound, which, with the aid of formula (77.6) of [10], may be written in the form

$$\Phi(t) = \exp\left\{-\frac{\Omega^2}{2c^2}\left[\frac{4}{3}\nu + \frac{\zeta}{\rho} + \chi(\gamma - 1)\right]t\right\} \quad \left(\gamma = \frac{c_p}{c_v}\right). \quad (2.27)$$

Here ζ is the second viscosity coefficient, χ is the diffusivity, and c_p and c_v are the specific heats. It is clear that (2.26) is a special case of (2.27) when $\zeta = 0$ and $\chi = 0$. Using this fact, we may at once, without further calculation, generalize (2.21) to the case in which $\zeta \neq 0$ and $\chi \neq 0$,

$$\Phi(t) = \exp\left\{-\frac{k_0^2}{2(1 - \beta^2)}\left[\frac{4}{3}\nu + \frac{\zeta}{\rho} + \chi(\gamma - 1)\right]t\right\}. \quad (2.28)$$

This is the basic law (neglecting oscillations and the weak power-law attenuation) by which the amplitude of perturbations decreases at the front of a shock wave, when the dissipation factors (viscosity and thermal conductivity) are sufficiently small. We note that the results obtained (formulas (2.21) and (2.28)) are not related to the form of the boundary conditions (2.4).

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